

On the \mathcal{NP} -hardness of GRACSIM DRAWING and κ -SEFE Problems

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Abstract. We study the complexity of two problems in simultaneous graph drawing. The first problem, GRACSIM DRAWING, asks for finding a simultaneous geometric embedding of two graphs such that only crossings at right angles are allowed. The second problem, κ -SEFE, is a restricted version of the topological simultaneous embedding with fixed edges (SEFE) problem, for two planar graphs, in which every private edge may receive at most k crossings, where k is a prescribed positive integer. We show that GRACSIM DRAWING is \mathcal{NP} -hard and that κ -SEFE is \mathcal{NP} -complete. The \mathcal{NP} -hardness of both problems is proved using two similar reductions from 3-PARTITION.

1 Introduction

The problem of computing a simultaneous embedding of two or more graphs has been extensively explored by the graph drawing community. Indeed, besides its inherent theoretical interest [1,2,4,5,6,7,9,10,11,12,13,14,15,16,17,18,19,22,23,24,25,26], it has several applications in dynamic network visualization, especially when a visual analysis of an evolving network is needed. Although many variants of this problem have been investigated so far, a general formulation for two graphs can be stated as follows: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two planar graphs sharing a *common* (or *shared*) subgraph $G = (V, E)$, where $V = V_1 \cap V_2$ and $E = E_1 \cap E_2$. *Compute a planar drawing Γ_1 of G_1 and a planar drawing Γ_2 of G_2 such that the restrictions to G of these drawings are identical.* By overlapping Γ_1 and Γ_2 in such a way that they perfectly coincide on G , it follows that edge crossings may only occur between a *private* edge of G_1 and a *private* edge of G_2 , where a *private* (or *exclusive*) edge of G_i is an edge of $E_i \setminus E$ ($i = 1, 2$).

Depending on the drawing model adopted for the edges, two main variants of the simultaneous embedding problem have been proposed: *topological* and *geometric*. The topological variant, known as SIMULTANEOUS EMBEDDING WITH FIXED EDGES (or SEFE for short), allows to draw the edges of Γ_1 and Γ_2 as arbitrary open Jordan curves, provided that every edge of G is represented by the same curve in Γ_1 and Γ_2 . Instead, the geometric variant, known as SIMULTANEOUS GEOMETRIC EMBEDDING (or SGE for short), imposes that Γ_1 and Γ_2 are two straight-line drawings. The SGE problem is therefore a restricted version of SEFE, and it turned out to be “too much restrictive”, i.e. there are examples of pairs of structurally simple graphs, such as a path and a tree [6], that do not admit an SGE. Also, testing whether two planar graphs admit a simultaneous

geometric embedding is \mathcal{NP} -hard [16]. Compared with SGE, pairs of graphs of much broader families always admit a SEFE, in particular there always exists a SEFE when the input graphs are a planar graph and a tree [18]. In contrast, it is a long-standing open problem to determine whether the existence of a SEFE can be tested in polynomial time or not, for two planar graphs; though, the testing problem is \mathcal{NP} -complete when generalizing SEFE to three or more graphs [22]. However, several polynomial time testing algorithms have been provided under different assumptions [3,4,11,12,24,26], most of them involve the connectivity or the maximum degree of the input graphs or of their common subgraph.

In this paper we study the complexity of the GEOMETRIC RAC SIMULTANEOUS DRAWING problem [8] (GRACSIM DRAWING for short): a restricted version of SGE, which asks for finding a simultaneous geometric embedding of two graphs, such that all edge crossings must occur at right angles. We show that GRACSIM DRAWING is \mathcal{NP} -hard by a reduction from 3-PARTITION; see Section 3. Moreover, we introduce a new restricted version of the SEFE problem, called κ -SEFE, in which every private edge may receive at most k crossings, where k is a prescribed positive integer. We then show that κ -SEFE is \mathcal{NP} -complete for any fixed positive k , to prove the \mathcal{NP} -hardness we use a similar reduction technique as that for GRACSIM DRAWING; see Section 4.

2 Preliminaries

Let $G = (V, E)$ be a simple graph. A *drawing* Γ of G maps each vertex of V to a distinct point in the plane and each edge of E to a simple Jordan curve connecting its end-vertices. Drawing Γ is *planar* if no two distinct edges intersect, except at common end-vertices. Γ is a *straight-line planar drawing* if it is planar and all its edges are represented by straight-line segments. G is *planar* if it admits a planar drawing. A planar drawing Γ of G partitions the plane into topologically connected regions called *faces*. The unbounded face is called the *external* (or *outer*) face; the other faces are the *internal* (or *inner*) faces. A face f is described by the circular ordering of vertices and edges that are encountered when walking along its boundary in clockwise direction if f is internal, and in counterclockwise direction if f is external. A *planar embedding* of a planar graph G is an equivalence class of planar drawings that define the same set of faces for G . A *plane graph* is a planar graph with an associated planar embedding and a prescribed outer face. Let H be a plane graph. The *weak dual* H^* of H is a graph whose vertices correspond to the internal faces of H , and there is an edge between two vertices if the corresponding internal faces in H share one or more edges. A *fan* is a graph formed by a path π plus a vertex v and a set of edges connecting v to every vertex of π ; vertex v is called the *apex* of the fan. A *wheel* is a graph consisting of a cycle C plus a vertex c and a set of edges connecting c to every vertex of C ; vertex c is the *center* of the wheel.

3 NP-hardness of GRACSIM DRAWING

In this section, we study the complexity of the following problem.

- Problem:** GEOMETRIC RAC SIMULTANEOUS DRAWING (GRACSIM DRAWING)
Instance: Two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ sharing a common subgraph $G = (V, E) = (V, E_1 \cap E_2)$.
Question: Are there two straight-line planar drawings Γ_1 and Γ_2 , of G_1 and G_2 , respectively, such that (i) every vertex is mapped to the same point in both drawings, and (ii) any two crossing edges e_1 and e_2 , with $e_1 \in E_1 \setminus E$ and $e_2 \in E_2 \setminus E$, cross only at right angle?

Theorem 1. *Deciding whether two graphs have a GRACSIM DRAWING is \mathcal{NP} -hard.*

Proof. We prove the \mathcal{NP} -hardness by a reduction from 3-PARTITION (3P).

- Problem:** 3-PARTITION (3P)
Instance: A positive integer B , and a multiset $A = \{a_1, a_2, \dots, a_{3m}\}$ of $3m$ natural numbers with $B/4 < a_i < B/2$ ($1 \leq i \leq 3m$).
Question: Can A be partitioned into m disjoint subsets A_1, A_2, \dots, A_m , such that each A_j ($1 \leq j \leq m$) contains exactly 3 elements of A , whose sum is B ?

We recall that 3P is a *strongly* \mathcal{NP} -hard problem [20], i.e., it remains \mathcal{NP} -hard even if B is bounded by a polynomial in m . Also, a trivial necessary condition for the existence of a solution is that $\sum_{i=1}^{3m} a_i = mB$, therefore it is not restrictive to consider only instances satisfying this equality.

We first give an overview of this reduction, then we describe in detail the construction for transforming an instance of 3P into an instance $\langle G_1, G_2 \rangle$ of GRACSIM DRAWING, and finally we prove that an instance of 3P is a *Yes*-instance if and only if the transformed instance $\langle G_1, G_2 \rangle$ admits a GRACSIM drawing.

OVERVIEW The transformed instance $\langle G_1, G_2 \rangle$ of GRACSIM DRAWING is obtained by combining a *subdivided pumpkin* gadget with $3m$ *subdivided slice* gadgets and m *transversal paths*; see Fig. 1 for an illustration. A *pumpkin* gadget consists of a biclique $K_{2,m+1}$ plus an additional edge, called the *handle*, that connects two vertices of the partite set of cardinality $m+1$; the two vertices of the other partite set are the *poles of the pumpkin*. A *subdivided pumpkin* is a *pumpkin* where each edge, other than the handle, is subdivided exactly once, while the handle is subdivided twice. We remark that it is not strictly necessary to use a subdivided pumpkin instead of a normal pumpkin, the only reason is to exploit the subdivision vertices as bend points, in this way we get more readable and compact GRACSIM drawings. Hereafter, when it is not ambiguous, we will use the terms *pumpkin* and *slice* in place of *subdivided pumpkin* and of *subdivided slice*, respectively. All the edges of a pumpkin are *shared* edges, that is, they belong to both graphs, therefore they cannot be crossed in any GRACSIM drawing. Moreover, any planar embedding of a subdivided pumpkin contains exactly two faces of degree seven and m faces of degree eight, the latter are called *wedges* and are the only faces

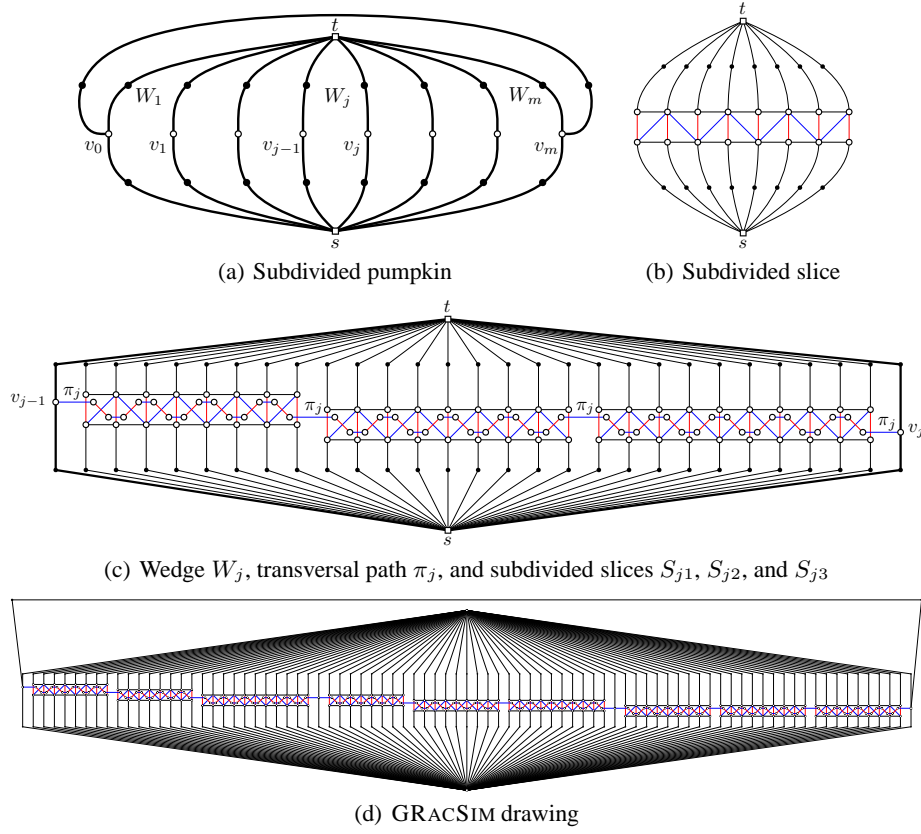


Fig. 1. Illustration of (a) a subdivided pumpkin gadget and of (b) a subdivided slice gadget encoding integer 7. (c) A wedge W_j of width 24, its transversal path π_j , and subdivided slices S_{j1} , S_{j2} , and S_{j3} encoding integers 7, 8 and 9, respectively. Shared edges are colored black, those of the subdivided pumpkin with thick lines, while private edges of G_1 and of G_2 are colored blue and red, respectively. (d) A (vertically stretched) GRACSIM drawing of the transformed instance of 3P, when $m = 3$, $B = 24$ and $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$. Subdivided slices are drawn within wedges according to the following solution of 3P: $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$.

incident to both poles. Wedges are used to contain (*subdivided*) *slice* gadgets, which are $3m$ subgraphs attached to the two poles of the pumpkin, with no other vertices in common with each other and with the pumpkin. Every slice has a “width” that suitably encodes a distinct element a_i of A —recall that two distinct elements could be equal—and the structure of a slice is sufficiently “rigid” so that overlaps and nestings among slices cannot occur in a GRACSIM drawing.

The basic idea of the reduction is to get the subsets A_j ($1 \leq j \leq m$) of a solution of 3P, in case one exists, by looking at the slices in each wedge of a GRACSIM drawing, which implies that every wedge must contain exactly three slices whose widths sum

to B . Of course, without introducing some further gadget, each wedge could contain even all slices, i.e. its width can be considered unlimited. Hence, in order to make all wedges of the same width B , m *transversal paths* are attached to the pumpkin, one for each wedge. Precisely, a *transversal path* is an alternating path that connects the two vertices of a wedge other than the poles and the subdivision vertices, and it contains only non-shared edges that belong alternatively to G_1 and to G_2 . Therefore, the pumpkin plus the transversal paths form a subdivision of a maximal planar graph, which has a unique embedding (up to a choice of the external face). Further, every transversal path has an “effective length” that encodes the integer B , which also establishes the width of the corresponding wedge. Crossings between slices and transversal paths are thus unavoidable in a GRACSIM drawing, because every transversal path splits its wedge into two parts, separating the two poles of the pumpkin; clearly, every slice crosses only one transversal path. However, by choosing a suitable structure for the slices, it is possible to form only crossings that are allowed in a GRACSIM drawing. The key factor of the reduction is to make it possible if and only if each slice of width a_i can cross a portion of its transversal path with an effective length greater than or equal to a_i . In other words, the slice structure and the transversal path effective length are defined in such a way that, in a GRACSIM drawing, (i) every transversal path cannot cross more than three slices, and (ii) the total width of the slices crossed by a same transversal path equals integer B , which yields a solution of 3P.

CONSTRUCTION We now describe in detail a procedure to incrementally construct an instance $\langle G_1, G_2 \rangle$ of GRACSIM DRAWING starting from an instance of 3P. At each step, this procedure adds one or more subgraphs (gadgets) to the current pair of graphs. As G_1 and G_2 have the same vertex set, for each added subgraph we will only specify which edges are shared and which are exclusive; the final vertex set will be known implicitly.

Start with a biclique $K_{2,m+1}$, and denote by s, t and by v_0, v_1, \dots, v_m its vertices of the partite sets of cardinality 2 and $m + 1$, respectively. Add edge $h = (v_0, v_m)$ to the biclique, subdivide h twice, and denote by π_h the resulting 3-edge path. Then, for every $0 \leq j \leq m$, subdivide edge (s, v_j) ((t, v_j) , respectively) exactly once, denote the subdivision vertex by v_j^s (v_j^t , respectively) and the 2-edge path obtained from this subdivision by $\pi_s(j)$ ($\pi_t(j)$, respectively). The resulting graph G_p is the *subdivided pumpkin* and all its edges are shared edges, i.e. $G_p \subset G$; vertices s and t are the *poles of the pumpkin*, while π_h is called the *subdivided handle* of the pumpkin.

Connect each pair of vertices v_{j-1}, v_j ($1 \leq j \leq m$) of G_p with a *transversal path* π_j , consisting of $2B + 1$ non-shared edges, so that edges in odd positions (starting from v_{j-1}) are private edges of G_1 , while those in even positions are private edges of G_2 ; hence, every transversal path starts and ends with an edge of G_1 and has exactly $2B$ inner vertices. Integer B represents the *effective length* of a transversal path, which is defined as half the number of its inner vertices.

For each integer $a_i \in A$, ($1 \leq i \leq 3m$) construct a (*subdivided*) *slice* S_i by suitably attaching two fan subgraphs and by subdividing a subset of their edges as follows (see, e.g., Fig.1(b)). Add a fan of $a_i + 2$ vertices with apex at pole t and subdivide every edge incident to t exactly once; denote the resulting subdivided fan by F_i^t . Specularly, add a subdivided fan F_i^s with apex at the other pole s , having the same number of ver-

tices as F_i^t . All the edges of F_i^s and F_i^t are shared edges, i.e. $F_i^s \cup F_i^t \subset G$. Now, let π_i^t and π_i^s be the two paths of these fans, i.e. $\pi_i^t = F_i^t \setminus \{t\}$ and $\pi_i^s = F_i^s \setminus \{s\}$. Visit path π_i^t starting from one of its end-vertices and denote the k -th encountered vertex by $\pi_i^t(k)$ ($1 \leq k \leq a_i + 1$); in an analogous way define the k -th vertex $\pi_i^s(k)$ of path π_i^s . For each $1 \leq k \leq a_i + 1$, connect $\pi_i^s(k)$ to $\pi_i^t(k)$ with a private edge of G_2 . Further, for each $1 \leq k \leq a_i$, add a private edge of G_1 joining either $\pi_i^s(k)$ to $\pi_i^t(k+1)$ or $\pi_i^t(k)$ to $\pi_i^s(k+1)$ depending on whether k is odd or even, respectively. We conclude this construction by introducing the concepts of *tunnel* and of *width* of a slice. The *tunnel* Δ_i is the subgraph of S_i induced by the vertices of π_i^t and π_i^s , i.e. $\Delta_i = S_i[V(\pi_i^s) \cup V(\pi_i^t)]$. It is straightforward to see that every tunnel is a biconnected internally-triangulated outer-plane graph, its weak dual is a path, and it contains exactly $2a_i$ triangles. The *width* $w(S_i)$ of a slice S_i is defined as half the number of triangles in its tunnel.

It is not difficult to see that the transformed instance of GRACSIM DRAWING contains $6Bm + 21m + 7$ vertices and $10Bm + 20m + 7$ edges, therefore its construction can be performed in polynomial time. We observe that the common subgraph is not connected. Indeed, G consists of the pumpkin G_p along with all fans and all inner vertices of the transversal paths; thus, there are $2Bm$ isolated vertices in the common subgraph. Moreover, even G_1 and G_2 are not connected, because in addition to G they also contain their own private edges of slices S_i ($1 \leq i \leq 3m$) and those of transversal paths π_j ($1 \leq j \leq m$); in particular, due to the latter paths, G_1 and G_2 contain an induced matching of $(B-1)m$ and Bm (private) edges, respectively.

CORRECTNESS We now prove that a *Yes*-instance of 3P is transformed into a *Yes*-instance of GRACSIM DRAWING, and vice-versa.

(\Rightarrow) Let A be a *Yes*-instance of 3P, we show how to compute a GRACSIM drawing of the transformed instance $\langle G_1, G_2 \rangle$ on an integer grid; it suffices to compute the vertex coordinates, because edges are represented by straight-line segments. The drawing construction strongly relies on the concepts of *square cell* and of *cell array*. A *square cell*, or briefly a *cell*, is a 4×4 square, with corners at grid points, and with opposite sides that are either horizontal or vertical. The diagonal of a cell connecting the bottom-left (top-left, respectively) and the top-right (bottom-right, respectively) corners is called the *positive-slope diagonal* (*negative-slope diagonal*). The *center* of a cell is the intersection point of its diagonals, which meet at right angles. Every cell contains four special grid points, called *anchor points*, which are the corners of a 2×2 square having the same center as the cell; two anchor points lie on the positive-slope diagonal while the other two are on the negative-slope diagonal. A *horizontal cell array* CA of length $l > 0$ is an ordered sequence c_1, c_2, \dots, c_l of l cells such that any two consecutive cells c_p, c_{p+1} ($1 \leq p < l$) share a vertical side; namely, the right side of c_p coincide with the left side of c_{p+1} .

Consider now a solution $\{A_1, A_2, \dots, A_m\}$ of 3P for the instance A . For each triple A_j ($1 \leq j \leq m$), denote its elements by a_{j1}, a_{j2}, a_{j3} , i.e. $A_j = \{a_{j1}, a_{j2}, a_{j3}\} \subset A$, and denote by S_{j1}, S_{j2} and S_{j3} , and by Δ_{j1}, Δ_{j2} and Δ_{j3} , the corresponding slices and their tunnels in the transformed instance. Embed each tunnel Δ_{jk} ($1 \leq k \leq 3$) on a horizontal array CA_{jk} of length a_{jk} in such a way that the private edges of G_2 are represented by the vertical sides of cells in CA_{jk} . The private edges of G_1 are

thus embedded on a sequence of a_{jk} cell diagonals, whose slopes are alternately $+1$ (positive-slope diagonal) and -1 (negative-slope diagonal), starting from $+1$; hence, in every cell, the anchor points of one of the two diagonals are *occupied*, i.e. they overlap with a straight-line segment representing a private edge of G_1 , while the remaining two anchor points are (still) *free*.

Place cell arrays CA_{jk} one after another, from left to right, in increasing order of $j = 1, 2, \dots, m$ and, in case of ties, in increasing order of $k = 1, 2, 3$. Also, leave a horizontal gap of one cell between intra-partition consecutive arrays and a horizontal gap of two cells in case of inter-partition consecutive arrays. Concerning the vertical placement proceed as follows. Let CA and CA' be two arbitrary consecutive arrays (intra- or inter-partition), with CA to the left of CA' . If CA has an even length, then CA and CA' are top- and bottom-aligned along the vertical axis, while if CA has an odd length, then CA' is shifted down of half a cell with respect to CA . It follows that the rightmost free anchor point of CA is always horizontally aligned with the leftmost free anchor point of CA' . Now, let R be the smallest rectangle containing all previous cell arrays with a top, right, bottom, and left margin of one cell. Place pole t (s , respectively) at a grid point above (under, respectively) the top side (bottom side, respectively) of R , as close as possible to its vertical bisector line, leaving a vertical offset of two cells; in Fig. 1(d) we deliberately increased this offset to get a better aspect ratio. Place vertex v_j ($0 \leq j < m$) at the grid point that is horizontally aligned with and to the left of the first free anchor point of CA_{j1} , leaving a margin of one cell; also, place vertex v_m at the grid point that is horizontally aligned with and one-cell to the right of the rightmost free anchor point. Observe that v_0 and v_m lie on the left and right side of R , respectively. Now, embed the vertices v_j^t and v_j^s ($j = 0, 1, \dots, m$) of the pumpkin G_p along the top and bottom side of R , respectively, in such a way that they are vertically aligned with v_j . Then, embed the missing vertices of the slices in an analogous way, that is a vertex adjacent to t (s , respectively) must be vertically aligned with its neighbor in the tunnel and must lie along the top side (bottom side, respectively) of R . Concerning the handle π_h , place its subdivision vertex adjacent to v_0 at the point whose x - and y -coordinates are one cell to the left of v_0 and one cell above t , respectively; with a symmetrical argument choose the position of the other subdivision vertex of π_h . It is not hard to see that (i) no crossing has been introduced so far; (ii) slices S_{j1} , S_{j2} and S_{j3} are within wedge W_j ($1 \leq j \leq m$); and (iii) every triangle in a tunnel contains exactly one free anchor vertex. To complete the drawing, it remains to embed the inner vertices of transversal paths, taking into account that every path π_j will unavoidably cross the three slices in its edge W_j . Place these vertices at the free anchor points, so that the p -th inner vertex of π_j occupies the p -th free anchor point, from left to right. It turns out that the produced crossings will always occur at right angles and involve a private edge of G_1 and a private edge of G_2 . Note that this is possible because, by construction, $w(W_j) = B = w(S_{j1}) + w(S_{j2}) + w(S_{j3})$, where $w(W_j)$ is the *width* of wedge W_j , which is defined as the effective length of π_j . Indeed, π_j has $2B$ inner vertices, there are $2(a_{j1} + a_{j2} + a_{j3})$ free anchor points in W_j , and $a_{j1} + a_{j2} + a_{j3} = B$, since we start from a solution of 3P.

(\Leftarrow) Let $\langle \Gamma_1, \Gamma_2 \rangle$ be any GRACSIM drawing of $\langle G_1, G_2 \rangle$, and let Γ_p be the drawing of G_p induced by $\langle \Gamma_1, \Gamma_2 \rangle$. Also, let $C_j \subset G_p$ ($1 \leq j \leq m$) be the cycle consisting of

paths $\pi_s(j-1)$, $\pi_t(j-1)$, $\pi_t(j)$ and $\pi_s(j)$. We first claim that the following invariants are satisfied. (I1) C_j ($1 \leq j \leq m$) is the boundary of a wedge W_j in Γ_p , where a wedge is a bounded or unbounded face of degree eight in Γ_p . (I2) Transversal path π_j ($1 \leq j \leq m$) is drawn within wedge W_j . (I3) Any two slices cannot be contained one in another and do not overlap with each other except at poles s and t . (I4) Every edge of π_j ($1 \leq j \leq m$) crosses at most one edge of a same slice. (I5) Every wedge contains exactly three slices.

Let $R_b(C_j)$ and $R_u(C_j)$ be the bounded and the unbounded plane regions, respectively, delimited by C_j in Γ_p . Since v_{j-1} and v_j are two vertices of C_j , path π_j has to be drawn within either $R_b(C_j)$ or $R_u(C_j)$, otherwise an inner edge of π_j would cross an edge of C_j , which is not allowed in a GRACSIM drawing of $\langle G_1, G_2 \rangle$ because $C_j \subset G$. Also, if π_j is contained in $R_b(C_j)$ ($R_u(C_j)$, respectively), then all the other paths of the pumpkin that connect the two poles s and t must be drawn within $R_u(C_j)$ ($R_b(C_j)$, respectively). Invariants I1 and I2 are thus satisfied. Concerning invariant I3, it is immediate to see that any two slices cannot be contained one in another. Further, in case of overlap, an edge e_1 of a slice S_1 would cross a boundary edge e_2 of a slice S_2 , where e_2 is a private edge of G_2 and e_1 is a private edge of G_1 . But this is not possible, because the end-vertices of e_1 are also connected in S_1 by a 2-edge path consisting of a shared edge and of a private edge of G_2 . Invariant I4 holds because every transversal path π_j ($1 \leq j \leq m$) can only cross edges of tunnels in W_j , and every tunnel is drawn as a straight-line internally triangulated outer-plane graph. Therefore, π_j cannot enter and then exit from a triangle with a same private edge in such a way that all edge crossings are at right angles. Namely, every triangle of a tunnel in W_j takes at least one inner vertex of π_j . We now show that invariant I5 is satisfied. It is straightforward to see that every slice must be drawn within some wedge, and all the slices in a wedge W_j are crossed by its transversal path π_j . In particular, π_j has to pass through the tunnels of these slices and such tunnels are pairwise disjoint and none of them contains another. Suppose by contradiction that invariant I5 does not hold. Then, there would be a wedge W_p ($1 \leq p \leq m$) containing at least four slices; recall that there are $3m$ slices to be distributed among m wedges. Let us denote such slices by $S_{p1}, S_{p2}, \dots, S_{pk}$, with $k \geq 4$, and let $a_{pl} \in A$ be the integer encoded by slice S_{pl} ($1 \leq l \leq k$). Since each element of A is strictly greater than $B/4$, it follows that $\sum_{l=1}^k w(S_{pl}) = \sum_{l=1}^k a_{pl} > \sum_{l=1}^k B/4 \geq B = w(W_p)$, thus wedge W_p is not wide enough to host all its slices, a contradiction. In other words, the alternating path π_p does not have enough inner vertices to pass through all the tunnels of slices in W_p avoiding crossing that are not allowed in a GRACSIM drawing.

Now, for each wedge W_j ($1 \leq j \leq m$), denote by S_{j1} , S_{j2} and S_{j3} the three slices that are within W_j , and let a_{j1} , a_{j2} and a_{j3} be their corresponding elements of A . We claim that $a_{j1} + a_{j2} + a_{j3} = B$. Indeed, it cannot be $\sum_{k=1}^3 a_{jk} > B$, because it would imply that $\sum_{k=1}^3 w(S_{jk}) > w(W_j)$, which is not possible as seen above. On the other hand, if $\sum_{k=1}^3 a_{jk} < B$, there would be some $j' \neq j$ with $1 \leq j' \leq m$ such that $\sum_{k=1}^3 a_{j'k} > B$, otherwise $\sum_{i=1}^{3m} a_i$ would be strictly less than mB , which violates our initial hypothesis on the elements of A . Hence, even this case is not possible. In conclusion, every wedge W_j ($1 \leq j \leq m$) contains exactly three slices S_{j1} , S_{j2} and S_{j3} , each of these slices has a width $w(S_{jk})$ ($1 \leq k \leq 3$) that encodes a distinct element of A , and the

sum of these widths is equal to B , i.e. $w(S_{j1}) + w(S_{j2}) + w(S_{j3}) = B$. Therefore, the partitioning of A defined by A_1, A_2, \dots, A_m , where $A_j = \{w(S_{j1}), w(S_{j2}), w(S_{j3})\}$, is a solution of 3P for the instance A . \square

We conclude this section with two remarks.

Remark 1. It is not hard to see that this reduction can also be used to give an alternative proof for the \mathcal{NP} -hardness of SGE, which was proved by Estrella-Balderrama et al. [16].

Remark 2. It is not clear whether this reduction can be adapted to study the complexity of the *one bend extension* of GRACSIM, i.e. the variant of GRACSIM in which one bend per edge is allowed; we leave this question as open problem.

4 \mathcal{NP} -completeness of κ -SEFE

In order to increase the readability of a simultaneous embedding, which is particularly desired in graph drawing applications, one may wonder whether it is possible to compute a SEFE, where every private edge receives at most a limited and fixed number of crossings. We recall that there is no restriction on the number of crossings that involve a private edge in a SEFE drawing. Further, two private edges may cross more than once, and these multiple crossings could be necessary for the existence of a simultaneous embedding; however, Frati et al. [19] have shown that whenever two planar graphs admit a SEFE, then they also admit a SEFE with at most sixteen crossings per edge pair.

Motivated by the previous considerations, we introduce and study the complexity of the following problem, named κ -SEFE, where k denotes a fixed bound on the number of crossings per edge that are allowed.

Problem: κ -SEFE

Instance: Two planar graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$, sharing a common subgraph $G = (V, E) = (V, E_1 \cap E_2)$, and a positive integer k .

Question: Do G_1 and G_2 admit a SEFE such that every private edge receives at most k crossings?

It is straightforward to see that κ -SEFE is, in general, a restricted version of SEFE. Namely, for any positive integer k , it is easy to find pairs of graphs that admit a $(\kappa+1)$ -SEFE, and thus a SEFE, but not a κ -SEFE. For example, consider a pair of graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ defined as follows (an illustration for $k = 4$ is given in Fig. 2). The common subgraph $G = (V, E)$ is a wheel of $2k + 5$ vertices, where $u_0, u_1, \dots, u_{k+1}, v_0, v_1, \dots, v_{k+1}$ are the $2(k + 2)$ vertices of its cycle in clockwise order, $E_1 = E \cup \{(u_0, v_0)\}$, and $E_2 = E \cup \bigcup_{i=1}^{k+1} \{(u_i, v_{k+2-i})\}$. Since G has a unique planar embedding (up to a homomorphism of the plane), the private edge (u_0, v_0) of G_1 crosses all the $k + 1$ private edges of G_2 , i.e. all the edges (u_i, v_{k+2-i}) with $1 \leq i \leq k + 1$. Therefore, G_1 and G_2 admit a $(\kappa+1)$ -SEFE, and thus a SEFE, but not a κ -SEFE.

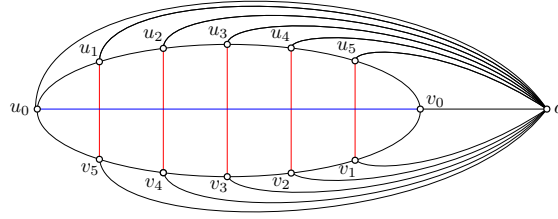


Fig. 2. A pair of graphs that admit a κ -SEFE only for $k \geq 5$.

Theorem 2. 1-SEFE is \mathcal{NP} -hard.

Proof. We use a reduction from 3P similar to that in the proof of Theorem 1; subdivision vertices are now omitted, since we are no longer in a geometric setting.

CONSTRUCTION Start with a (non-subdivided) pumpkin $G_p \subset G$ whose vertices v_0, v_1, \dots, v_m are adjacent to the two poles s and t , and whose handle is a single edge (v_0, v_m) . Add a transversal path π_j between every pair of vertices v_{j-1} and v_j ($1 \leq j \leq m$). Differently from the proof of Theorem 1, π_j has to contain $2B - 1$ inner vertices and not $2B$; the reason of this will be clarified later. Also, the effective length of π_j is now defined as half the number of its edges, hence it is still equal to B . Slice gadgets S_i ($1 \leq i \leq 3m$) and their tunnels Δ_i are also slightly modified and are defined as follows. For each integer $a_i \in A$, create an alternating path $\pi(S_i)$ of $2a_i$ non-shared edges; thus, $\pi(S_i)$ has $2a_i + 1$ vertices and its extremal edges never belong to the same graph G_i ($i = 1, 2$). Construct a fan F_i^t by adding an edge between all the pairs of consecutive vertices of $\pi(S_i)$ in even positions and by connecting such vertices to the pole t of the pumpkin; $F_i^t \setminus \{t\}$ is a path of $a_i - 1$ edges, because $\pi(S_i)$ has a_i vertices in even positions and $a_i + 1$ vertices in odd positions. Similarly, construct a fan F_i^s by connecting the pole s with a path of a_i edges passing through all the vertices of $\pi(S_i)$ in odd positions. Slice S_i is composed from the two fans F_i^t and F_i^s plus all the edges of $\pi(S_i)$. Further, all the edges of fans are shared, while those of $\pi(S_i)$ are not shared and belong alternatively to G_2 and to G_1 . The tunnel Δ_i of a slice S_i is the subgraph that results from S_i after removing the two poles s and t , i.e. $\Delta_i = S_i \setminus \{s, t\}$. It is straightforward to see that every tunnel is a biconnected internally-triangulated outer-plane graph, whose weak dual is a path, and it contains exactly $2a_i - 1$ triangles if the corresponding slice encodes integer a_i . The width $w(S_i)$ of a slice S_i is defined as half the number of private edges in its tunnel Δ_i , thus $w(S_i) = a_i$.

It is not hard to see that the transformed instance $\langle G_1, G_2 \rangle$ contains $4Bm + 9m + 3$ vertices and $8Bm + 2m + 3$ edges, thus its construction can be done in polynomial time. Furthermore, we observe that G, G_1 and G_2 are not connected. Indeed, G contains $(2B - 1)m$ isolated vertices, i.e. all the inner vertices of transversal paths, while G_1 and G_2 contain an induced matching of $(B - 1)m$ (private) edges each.

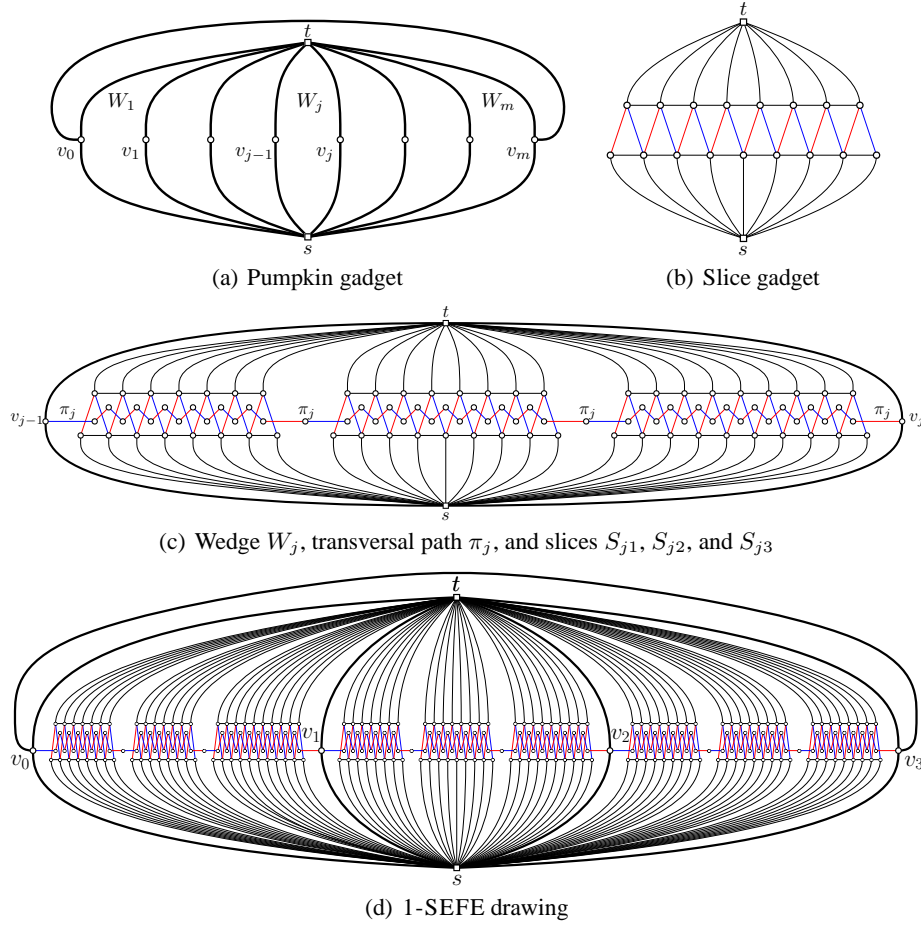


Fig. 3. Illustration of (a) a pumpkin gadget and of (b) a slice gadget encoding integer 8. (c) A wedge W_j of width 24, its transversal path π_j , and slices S_{j1} , S_{j2} , and S_{j3} encoding integers 7, 8 and 9, respectively. Shared edges are colored black, those of the pumpkin with thick lines, while private edges of G_1 and of G_2 are colored blue and red, respectively. (d) A 1-SEFE drawing of the transformed instance of 3P, when $m = 3$, $B = 24$ and $A = \{7, 7, 7, 8, 8, 8, 8, 9, 10\}$. Slices are drawn within wedges according to the following solution of 3P: $A_1 = \{7, 7, 10\}$, $A_2 = \{7, 8, 9\}$ and $A_3 = \{8, 8, 8\}$.

CORRECTNESS Let A be an instance of 3P, and let $\langle G_1, G_2 \rangle$ be an instance of 1-SEFE obtained by using the previous transformation. We show that A admits a 3-partition if and only if $\langle G_1, G_2 \rangle$ admits a 1-SEFE drawing.

(\Rightarrow) Suppose that A admits a 3-partition $\{A_1, A_2, \dots, A_m\}$, then a 1-SEFE drawing of $\langle G_1, G_2 \rangle$ can be constructed as follows. Compute a plane drawing Γ_p of the pumpkin G_p (see, e.g., Fig. 3(a)) such that (i) the external face is delimited by the edges (s, v_0) , (v_0, v_m) and (v_m, s) and (ii) for each $j = 1, 2, \dots, m$ edge (t, v_j) immediately

follows edge (t, v_{j-1}) in the counterclockwise edge ordering around t . Γ_p contains m inner faces of degree four, delimited by edges $(s, v_{j-1}), (v_{j-1}, t), (t, v_j), (v_j, s)$ ($1 \leq j \leq m$), which are the wedges W_j of the pumpkin. Consider now each triple $A_j = \{a_{j1}, a_{j2}, a_{j3}\}$ ($1 \leq j \leq m$), and denote by S_{j1}, S_{j2}, S_{j3} the corresponding slices in the transformed instance. For each slice S_{jk} ($1 \leq k \leq 3$), compute a plane drawing with both poles on the external face. Place these drawings one next to the other within wedge W_j , in any order; for simplicity we may assume that S_{j1} is the leftmost slice, S_{j2} is the middle slice and S_{j3} is the rightmost one. Also, if necessary, flip each slice around its poles so that the leftmost private edge always belongs to G_2 ; clearly, this implies that the rightmost private edge belongs to G_1 . It is not difficult to see that the drawing produced so far is planar, i.e. even the private edges do not create crossings. Moreover, since $w(W_j) = B = a_{j1} + a_{j2} + a_{j3} = w(S_{j1}) + w(S_{j2}) + w(S_{j3})$, every transversal path π_j ($1 \leq j \leq m$) can be drawn within wedge W_j in such a way that (i) every edge of π_j crosses exactly one private edge of a tunnel in W_j , and (ii) every crossing involves a private edge of G_1 and a private edge of G_2 .

(\Leftarrow) We conclude the proof by showing that if $\langle G_1, G_2 \rangle$ admits a 1-SEFE drawing $\langle \Gamma_1, \Gamma_2 \rangle$, then A admits a 3-partition. By a similar argument as that in the proof of Theorem 1, $\langle \Gamma_1, \Gamma_2 \rangle$ induces a plane drawing Γ_p of the pumpkin G_p , in which each wedge W_j , i.e. each bounded or unbounded face of degree four of G_p , is delimited by a cycle C_j consisting of edges $(s, v_{j-1}), (v_{j-1}, t), (t, v_j)$ and (v_j, s) , for some $1 \leq j \leq m$. Further, path π_j has to be drawn within W_j , and for each $1 \leq i \leq 3m$, fans F_i^t and F_i^s , and thus the slice S_i they belong to must be placed within a same wedge. Let $S_{j1}, S_{j2}, \dots, S_{jk}$ be the slices within wedge W_j , for some $k \geq 0$. Since every private edge receives at most one ($k = 1$) crossing in $\langle \Gamma_1, \Gamma_2 \rangle$, it follows that $\sum_{l=1}^k w(S_{jl}) \leq w(W_j) = B$, i.e. the number of edges of π_j must be greater than or equal to the number of edges of tunnels in W_j . We now show that there are exactly three slices in every wedge, i.e. $k = 3$. It cannot be $k > 3$, otherwise $\sum_{l=1}^k w(S_{jl}) = \sum_{l=1}^k a_{jl} > \sum_{l=1}^k B/4 \geq B = w(W_j)$. On the other hand, it cannot be $k < 3$, otherwise there would some other wedge with $k' > 3$ slices; recall that there are a total of $3m$ slices and a total of m wedges. Suppose now that $\sum_{l=1}^3 w(S_{jl}) < w(W_j) = B$, for some $1 \leq j \leq m$. Then, there would exist some $j' \neq j$ with $1 \leq j' \leq m$ such that $\sum_{l=1}^3 w(S_{j'l}) > w(W_{j'}) = B$, otherwise it would be violated the equality $\sum_{i=1}^{3m} a_i = mB$. In conclusion, there are exactly three slices in every wedge, and the sum of their widths coincides with B . Therefore the partitioning A_1, A_2, \dots, A_m of A , where $A_j = \{w(S_{j1}), w(S_{j2}), w(S_{j3})\}$, is a 3-partition. \square

Theorem 3. For any fixed $k \geq 1$, κ -SEFE is \mathcal{NP} -complete.

Proof. Concerning the \mathcal{NP} -hardness, it suffices to repeat the proof of Theorem 2, by replacing every private edge e of each tunnel of G_i ($i = 1, 2$) with a set of k internally vertex-disjoint paths $\pi_1(e), \pi_2(e), \dots, \pi_k(e)$, consisting each one of two private edges of G_i .

We now introduce some definitions and then prove the membership in \mathcal{NP} using an approach similar to that described in [21]. An *edge crossing structure* $\chi(e_1)$ of a private edge $e_1 \in E_1$ is a pair $\langle \varepsilon_2, \sigma(\varepsilon_2) \rangle$, where ε_2 is a multiset on the set $E_2 \setminus$

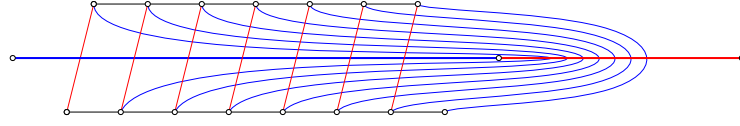


Fig. 4. Illustration of the 2-edge penetration vulnerability.

E with cardinality at most k , and $\sigma(\varepsilon_2)$ is a permutation of multiset ε_2 . A *crossing structure* $\chi(G_1, G_2)$ of a pair of graphs $\langle G_1, G_2 \rangle$ is an assignment of an edge crossing structure to each private edge of E_1 . Of course, all crossing structures of $\langle G_1, G_2 \rangle$ can be non-deterministically generated in a time that is polynomial in $|V| = n$, and they include the crossing structures induced by all κ -SEFE drawings of $\langle G_1, G_2 \rangle$. We conclude the proof by describing a polynomial time algorithm for testing whether a given crossing structure $\chi(G_1, G_2)$ is a crossing structure induced by some κ -SEFE drawing of $\langle G_1, G_2 \rangle$. Let G_U be the union graph of G_1 and G_2 , i.e. $G_U = (V, E_1 \cup E_2)$. For each edge e of G_U such that $e \in E_1 \setminus E$, consider its crossing structure $\chi(e) = \langle \varepsilon_2, \sigma(\varepsilon_2) \rangle$, replace every crossing between e and the edges in ε_2 with a dummy vertex, preserving the ordering given by $\sigma(\varepsilon_2)$, and then test the resulting (multi) graph for planarity. \square

We conclude even this section with two remarks.

Remark 3. The previous reduction cannot be successfully applied to SEFE, because of the *2-edge penetration vulnerability*: every transversal path π_j ($1 \leq j \leq m$) can pass through all the tunnels in W_j using only its two first edges; an illustration of this vulnerability is given in Fig. 4. Also, any tentative to patch this vulnerability by replacing the transversal paths with different graphs, modifying the slices accordingly, always resulted in constructions in which overlapping slices were possible.

Remark 4. From a theoretical point of view, it also makes sense to study a slightly different restriction of SEFE, where instead of limiting the number of crossings per edge, it is limited the number of distinct edges that cross a same private edge; recall that two private edges may cross each other more than once, which gives rise to a different problem than κ -SEFE. We may call this problem κ -PAIR-SEFE, because k is now the bound on the allowed number of crossing edge pairs involving a same edge. It is not hard to see that a reduction analogous to that given in the proof of Theorems 2 and 3 can be used to prove the \mathcal{NP} -hardness of κ -PAIR-SEFE. The interesting theoretical aspect of κ -PAIR-SEFE is the following: if k is greater than or equal to the maximum number of edges of G_i ($i = 1, 2$), then a κ -PAIR-SEFE is also a SEFE; in particular, if $k \geq 3|V| - 6$ the two problems are identical.

5 Conclusions and Open Problems

In this work we have shown the \mathcal{NP} -hardness of the GRACSIM DRAWING problem, a restricted version of the SGE problem in which edge crossings must occur only at right angles. Then, we have introduced and studied the \mathcal{NP} -completeness of the κ -SEFE problem, a restricted version of the SEFE problem, where every private can receive at most k crossings.

Our results raise two main questions. First, as already mentioned at the end of Section 3, it would be interesting to study the complexity of a relaxed version of the GRACSIM DRAWING problem, where a prescribed number of bends per edge are allowed; this open problem was already posed in [9]. In particular, it is not clear whether the reduction given in the proof of Theorem 1 can be adapted for proving the \mathcal{NP} -hardness of the one bend extension of GRACSIM. Another interesting open problem is to investigate the complexity of κ -PAIR-SEFE when the ratio $|V|/k$ tends to $\frac{1}{3} + \frac{2}{k}$ from the right; we recall that for $k \geq 3|V| - 6$, κ -PAIR-SEFE and SEFE are the same problem, and that the \mathcal{NP} -hardness of κ -PAIR-SEFE strongly relies on a construction where the ratio $|V|/k$ is significantly greater than $\frac{1}{3} + \frac{2}{k}$.

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